Solution : (i) Let X be a random variable, then

$$\sum_{i=0}^{6} p(x_i) = 1 \quad \text{or,} \quad k + 3k + 5k + 7k + 9k + 11k + 13k = 1 \quad \text{or,} \quad 49k = 1 \quad \text{i.e.,} \quad k = \frac{1}{49}.$$

$$P(X < 4) = k + 3k + 5k + 7k = 16k \quad \text{or,} \quad P(X < 4) = 16 \times \frac{1}{49} = \frac{16}{49}.$$

$$P(X \geq 5) = 11k + 13k = 24k = \frac{24}{49}$$

and $P(3 < X \leq 6) = 9k + 11k + 13k = 33k = \dfrac{33}{49}$.

(ii) $P(X \leq 2) = k + 3k + 5k = 9k > 0.3$.

23.12 Theoretical Probability Distribution

The probability distribution of many variables of interest can be developed from purely theoretical considerations. Some of such distributions having wide range of applications are as below :

1. Binomial distribution, 2. Poisson distribution and 3. Normal distribution.

23.12.1 Binomial Distribution

Binomial distribution concerned with trial of a repetitive nature in which only the occurrence or non-occurrence, success or failure, acceptance or rejection, yes or no of a particular event is of interest. It has wide applications in statistics. This is also known as **Bernoulli's distribution.**

Expression for Binomial distribution $P(r)$: This means to find the probability of happening of an event once, twice, thrice, ..., r times exactly in $n(n > r)$ number of trials.

Suppose, the probability of happening an event A in one trial be p and the probability of not happening be $1 - p = q$. As there are n trials and the probability of not happening is $n - r$ times, we may represent the situation as :

$$\underbrace{AAA \cdots\cdots}_{r\text{-times}} \underbrace{\overline{A}\,\overline{A}\,\overline{A} \cdots\cdots\cdots}_{(n-r)\text{-times}} \overline{A}\,\overline{A}\,, \tag{23.27}$$

where A and $\overline{A}$ represent respectively happening and not happening respectively.

Now, the order of arrangement in (23.27) has the probability

$$\underbrace{ppp \cdots\cdots}_{r\text{-times}} \underbrace{pqq \cdots\cdots\cdots}_{(n-r)\text{-times}} qp = p^r q^{n-r}. \tag{23.28}$$

Now, (23.27) is only one order of arrangement rA's.

The probability of (23.27) $= p^r q^{n-r} \times$ number of different arrangements of rA's.

The number of different arrangements of rA's and $(n - r)\overline{A}$'s $= {}^nC_r$.

$\therefore$ probability of happening of an event r-times $= {}^nC_r p^r q^{n-r}$

or, $P(r) = {}^nC_r p^r q^{n-r} \quad [r = 0, 1, 2, \ldots, n]$

i.e., $\boldsymbol{P(r) = (r+1)\text{th term of } (q+p)^n.}$ (23.29)

Eqn. (23.29) gives the **binomial distribution.**

Special Cases :

(i) If $r = 0$, then $P(0) = {}^nC_0 p^0 q^n = q^n$.

Thus the probability of happening an event A 0-times in n trial is q^n, the outcome being a failure.

(ii) If $r = 1$, $P(1) = {}^nC_1 pq^{n-1}$ = the probability of happening A in 1 time.

(iii) If $r = 2$, $P(2) = {}^nC_2 p^2 q^{n-2}$ = the probability of happening A 2 times.

(iv) If $r = 3$, $P(3) = {}^nC_3 p^3 q^{n-3}$ = the probability of happening A 3 times.

The above terms are the successive terms in the expansion $(p+q)^n$ and hence the name **binomial distribution.**

Example 23.17. *If 10% of bolts produced by a machine are defective, determine the probability that out of 10 bolts chosen at random* (i) 1 (ii) none (iii) at most two bolts will be defective. **[AMIETE(Winter) 1989, 1988]**

Solution : By the problem, probability of defective bolts $= p = 10\% = 0.1$.

Probability of undefective bolts $= q = 1 - p = 1 - 0.1 = 0.9$.

Total number of bolts, $n = 10$.

(i) Probability of one defective bolt $= {}^{10}C_1 (0.1)^1 (0.1)^9 =$ **0.3874.**

(ii) Probability that none is defective = probability of 0 defective bolts $= P(0)$

$= {}^{10}C_0 (0.1)^0 (0.9)^{10} =$ **3487.**

(iii) Probability of 2 defective bolts $= {}^{10}C_2 (0.1)^2 (0.9)^8 =$ **0.1937.**

Now, probability of at most 2 defective bolts $= P(0 \text{ or } 1 \text{ or } 2) = P(0) + P(1) + P(2)$

$= 0.3487 + 0.3874 + 0.1937$

$=$ **0.9298.**

23.12.2 Mean Binomial Distribution

For the binomial distribution

$$P(r) = {}^nC_r p^r q^{n-r},$$

where r = the number of times in which the success occurs, f = the corresponding frequency.

Successes (r)	Frequency (f)	Product (rf)
0	q^n	0
1	npq^{n-1}	npq^{n-1}
2	$\frac{n(n-1)}{2} p^2 q^{n-2}$	$n(n-1)p^2 q^{n-2}$
...	...	...
n	p^n	np^n

$$\therefore \quad \sum fr = npq^{n-1} + n(n-1)p^2 q^{n-2} + \frac{n(n-1)(n-2)}{2!} p^3 q^{n-3} + \cdots + np^n$$

$$= np\left[q^{n-1} + \frac{(n-1)}{1!} pq^{n-2} + \frac{(n-1)(n-2)}{2!} p^2 q^{n-3} + \cdots + p^{n-1}\right] = np(q+p)^{n-1}$$

$$= np, \quad [\because \ q + p = 1]$$

and $\sum f = q^n + npq^{n-1} + \frac{n(n-1)}{2!}p^2q^{n-2} + \cdots + p^n = (q+p)^n = 1.$

$\therefore$ **Mean binomial distribution,** $\mu = \dfrac{\sum fr}{\sum f} = np.$ (23.30)

23.12.3 Standard Deviation of Binomial Distribution

To derive the standard deviation (σ) of binomial distribution we refer to the following table :

Successes (r)	Frequency (f)	($r^2 f$) value
0	q^n	0
1	npq^{n-1}	npq^{n-1}
2	$\frac{n(n-1)}{2!}p^2q^{n-2}$	$2n(n-1)p^2q^{n-2}$
...	...	...
n	p^n	n^2p^n

$\therefore$ total frequency $= \sum f = 1.$

$$\sum r^2 f = npq^{n-1} + 2n(n-1)p^2q^{n-2} + \frac{3n(n-1)(n-2)}{2!}p^3q^{n-3} + \cdots + n^2p^n$$

$$= np\left[q^{n-1} + \frac{2n(n-1)}{1!}pq^{n-2} + \frac{3n(n-1)(n-2)}{2!}p^2q^{n-3} + \cdots + np^{n-1}\right]$$

$$= np\left[q^{n-1} + \frac{(n-1)pq^{n-2}}{1!} + \frac{(n-1)(n-2)}{2!}p^2q^{n-3} + \cdots + p^{n-1}\right.$$

$$\left. + \frac{(n-1)pq^{n-2}}{1!} + \frac{2(n-1)(n-2)}{2!}p^2q^{n-3} + \cdots + (n-1)p^{n-1}\right]$$

$$= np\left[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{2!}p^2q^{n-3} + \cdots + p^{n-1}\right.$$

$$\left. + (n-1)p\left\{q^{n-2} + (n-2)pq^{n-3} + \frac{(n-2)(n-3)}{2!}p^2q^{n-4} + \cdots + p^{n-2}\right\}\right]$$

$$= np[(p+q)^{n-1} + (n-1)p(p+q)^{n-2}] = np[1 + (n-1)p] = np[np + (1-p)]$$

$$= np[np + q] \quad [\because\ 1 - p = q]$$

$$= \boldsymbol{n^2p^2 + npq}$$

$\therefore$ **Variance,** $\sigma^2 = \dfrac{\sum r^2 f}{\sum f} - \left(\dfrac{\sum rf}{\sum f}\right)^2 = n^2p^2 + npq - (np)^2 = \boldsymbol{npq}.$ (23.31)

$\therefore$ **Standard deviation,** $\boldsymbol{\sigma = \sqrt{npq}}.$ (23.32)

Thus, for the binomial distribution, Mean $= np$

$\mu_2 = \sigma^2 = npq.$

Likewise, the following moment generating functions can be obtained :

$\mu_3 = npq(q-p)$

$\mu_4 = npq[1+3pq(n-2)]$

Here, the moment coefficient of skewness is

$$\beta_1 = \frac{3^2}{\mu_2^3} = \frac{[npq(q-p)]^2}{(npq)^3} = \frac{(q-p)^2}{npq}.$$

$$r_1 = \sqrt{\beta_1} = \frac{q-p}{\sqrt{npq}}.$$

Coefficient of Kurtosis is given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq[1+3pq(n-2)]}{(npq)^2} = 3 + \frac{1-6pq}{npq}.$$

$$r_2 = \beta_2 - 3 = \frac{1-6pq}{npq}.$$

Example 23.18. *A coin is tossed thrice. A success is getting* 1 *or* 6 *on a toss. Find the mean and variance of the number of success.* **[AMIE(Summer) 1991]**

Solution : Here, $n = 3$, $p = \frac{1}{3}$, $q = \frac{2}{3}$.

Mean $= np = 3 \times \frac{1}{3} = 1$, Variance $\sigma^2 = npq = 3 \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{3}$.

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{\frac{2}{3}}$.

23.13 Poisson Distribution

Poisson's distribution is related to the probability of events which are extremely rare, but which have a large number of independent opportunities for occurrence.

In physical problem like **radioactive disintegration**, this distribution finds important application.

Thus, in Poisson's distribution the probability of success $p \to 0$ and the number of trials $n \to \infty$ and np remains finite. With the above consideration, Poisson's distribution can be derived as a limiting case of binomial distribution.

The probability of r successes in binomial distribution is given by

$$P(r) = {}^nC_r p^r q^{n-r} = \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} \quad [\because\ p+q=1]$$

$$= \frac{1}{r!}[(n-1)(n-2)\cdots(n-r+1)] \left(\frac{m}{n}\right)^r \left(1-\frac{m}{n}\right)^{n-r}$$

[where np has been replaced by m, and p by $\left(\frac{m}{n}\right)$]

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)m^r}{n\cdot n\cdot n\cdot n\cdots nr!} \left(1-\frac{m}{n}\right)^n \left(1-\frac{m}{n}\right)^{-r}$$

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)\left(1-\frac{m}{n}\right)^n \left(1-\frac{m}{n}\right)^{-r} \frac{m^r}{r!}.$$

If now $n \to \infty$, then

$$P(r) = (1-0)(1-0)\cdots(1-0)e^{-m} \times (1-0)^{-r}\frac{m^r}{r!}$$

or, $$P(r) = \frac{m^r e^{-m}}{r!} \quad \left[\because \operatorname{Lt}_{n\to\infty}\left(1-\frac{m}{n}\right)^n = e^{-m}\right] \tag{23.32}$$

Eqn. (23.32) gives the **Poisson's distribution.**

Poisson's distribution may be defined as :

If x be the number of occurrences of some of events in an interval of time and space, probability that x will occur is

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where λ is a parameter being the number of occurrences of the random events in the interval.

N.B. From eqn. (23.32), the probabilities of 0, 1, 2, ..., r, ... successes in a Poisson's distribution are :

$$P(0) = e^{-m} \quad [\text{for } r = 0]$$
$$P(1) = me^{-m} \quad [\text{for } r = 1]$$
$$P(2) = \frac{m^2 e^{-m}}{2!} \quad [\text{for } r = 2]$$
$$P(3) = \frac{m^3 e^{-m}}{3!} \quad [\text{for } r = 3]$$
$$\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots$$
$$P(r) = \frac{m^r e^{-m}}{r!} \quad [\text{for } r = n]$$

23.13.1 Mean and Standard Deviation of Poisson's Distribution

We have $P(r) = \dfrac{e^{-m}m^r}{r!}$.

Successes (r)	Frequency (f)	Product ($f \cdot r$)
0	$\dfrac{e^{-m}m^0}{0!}$	0
1	$\dfrac{e^{-m}m^1}{1!}$	$e^{-m} \cdot m$
2	$\dfrac{e^{-m}m^2}{2!}$	$e^{-m} \cdot m^2$
3	$\dfrac{e^{-m}m^3}{3!}$	$\dfrac{e^{-m} \cdot m^3}{2!}$
...	...	...
r	$\dfrac{e^{-m}m^r}{r!}$	$\dfrac{e^{-m} \cdot m^r}{(r-1)!}$

$\therefore$ mean of Poisson's distribution is given by

$$\mu = \sum pr = p_1r_1 + p_2r_2 + p_3r_3 + \cdots$$

$$= e^{-m} \cdot 0 + me^{-m} \cdot 1 + \frac{m^2}{2!}e^{-m} \cdot 2 + \frac{m^3}{3!}e^{-m} \cdot 3 + \cdots + e^{-m}\frac{m^r}{(r-1)!} + \cdots$$

$$= e^{-m} \cdot m\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots + \frac{m^{r-1}}{(r-1)!} + \cdots\right] = me^{-m}[e^m] = m.$$

$\therefore$ **Mean $\mu = m$.** (23.34)

The variance of Poisson's distribution is $\sigma^2 = \sum pr^2 - \mu^2$.

Here, $\sum pr^2 = p_1r_1^2 + p_2r_2^2 + p_3r_3^2 + \cdots = e^{-m} \cdot 0 + me^{-m} \cdot 1^2 + \frac{m^2}{2!}e^{-m} \cdot 2^2 + \frac{m^3}{3!}e^{-m} \cdot 3^2 + \cdots$

$$= me^{-m}\left[1 + 2m + \frac{3m^2}{2!} + \frac{4m^3}{3^2} + \cdots\right]$$

$$= me^{-m}\left[\left(1 + m + \frac{m^2}{2!} + \cdots\right) + \left(m + \frac{2m^2}{2!} + \frac{3m^3}{3!} + \cdots\right)\right]$$

$$= me^{-m}\left[e^m + m\left(1 + m + \frac{2m^2}{2!} + \cdots\right)\right] = me^{-m}[e^m + me^m] = m(1+m).$$

$\therefore$ $\sigma^2 = m(1+m) - m^2 = \sum pr^2 - \sum pr^2 = m$ (23.35)

The standard deviation of Poisson's distribution is **S.D. $= \sigma = \sqrt{m}$** (23.36)

23.13.2 In Poisson's Distribution with Unit Mean and Mean Deviation about the Mean is $\left(\frac{2}{e}\right)$ × standard deviation (S.D.)

Proof : We have $P(r) = \frac{m^r}{r!}e^{-m}$.

But, mean $= 1$, i.e., $m = 1$ and S.D. $= \sqrt{m} = 1$.

$\therefore$ $P(r) = \frac{e^{-1}}{r!} = \frac{1}{e} \cdot \frac{1}{r!}$.

(r)	$P(r)$	$\lvert r-1\rvert$	$P(r)\lvert r-1\rvert$	(r)	$P(r)$	$\lvert r-1\rvert$	$P(r)\lvert r-1\rvert$
0	$\frac{1}{e}$	1	$\frac{1}{e}$	4	$\frac{1}{e}\frac{1}{4!}$	3	$\frac{1}{e}\frac{3}{4!}$
1	$\frac{1}{e}$	0	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$
2	$\frac{1}{e}\frac{1}{2!}$	1	$\frac{1}{e}\frac{1}{2!}$	r	$\frac{1}{e}\frac{1}{r!}$	$r-1$	$\frac{1}{e}\frac{r-1}{r!}$
3	$\frac{1}{e}\frac{1}{3!}$	2	$\frac{1}{e}\frac{2}{3!}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

$$\therefore \sum P(r)|r-1| = \frac{1}{e} + 0 + \frac{1}{e}\frac{1}{2!} + \frac{1}{e}\frac{2}{3!} + \frac{1}{e}\frac{3}{4!} + \cdots + \frac{1}{e}\frac{r-1}{r!} + \cdots$$

$$= \frac{1}{e}\left[1 + 0 + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{r-1}{r!} + \cdots\right]$$

$$= \frac{1}{e}\left[1 + \left(\frac{1}{1!} - \frac{1}{1!}\right) + \left(\frac{2}{2!} - \frac{1}{2!}\right) + \left(\frac{3}{3!} - \frac{1}{3!}\right) + \left(\frac{4}{4!} - \frac{1}{4!}\right) + \cdots + \left(\frac{r}{r!} - \frac{1}{r!}\right) + \cdots\right]$$

$$= \frac{1}{e}\left[1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \frac{4}{4!} + \cdots + \frac{r}{r!} + \cdots + \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \frac{1}{4!} \cdots - \frac{1}{r!} \cdots\right]$$

$$= \frac{1}{e}\left[1 + \left\{1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \frac{4}{4!} + \cdots + \frac{1}{(r-1)!} + \cdots\right\} - \left\{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{r!} + \cdots\right\} + 1\right]$$

$$= \frac{1}{e}[1 + e - e + 1] = \frac{2}{e} = \frac{2}{e}(1) = \frac{2}{e} \times \text{(S.D.)}. \tag{23.37}$$

23.13.3 Recurrence Formula for Poisson's Distribution

We know that in Poisson's distribution,

$$P(r) = \frac{e^{-m}m^r}{r!}. \tag{i}$$

$$\therefore\ P(r+1) = \frac{e^{-m}m^{r+1}}{(r+1)!}. \tag{ii}$$

Dividing Eqn. (ii) by Eqn. (i), we get

$$\frac{P(r+1)}{P(r)} = \frac{e^{-m}m^{r+1}}{(r+1)!} \cdot \frac{r!}{e^{-m}m^r} = \frac{m}{r+1}.$$

$$\therefore\ \boldsymbol{P(r+1) = \frac{m}{r+1}P(r).} \tag{23.38}$$

Example 23.19. *In a certain factory producing cycle tyres there is a small chance of* 1 *in* 500 *tyres to be defective. The tyres are supplied in lots of* 10. *Using Poisson's distribution, calculate the approximate number of lots containing* (i) no defective; (ii) one defective; (iii) two defective tyres respectively, in consignment of 10,000 lots.

[AMIETE(Summer) 1994; AMIE(Summer) 1994; WBUT 2004]

Solution : Here, $p = \frac{1}{500}$, $n = 10$, $m = np = 10 \times \frac{1}{500}$ or, $m = \frac{1}{50} = 0.02$.

$$\therefore\ P(r) = \frac{e^{-m}m^r}{r!}.$$

(i) Probability of no defective tyres, i.e., $P(0) = \frac{e^{-0.02}(0.02)^0}{0!} = e^{-0.02} = \mathbf{0.9802}$.

Number of lots containing no defective tyres $= 10,000 \times 0.9802 = \mathbf{9802}$ lots.

(ii) Probability of one defective tyre $P(1) = \frac{e^{-0.02}(0.02)}{1!} = 0.9802 \times 0.02 = \mathbf{0.019604}$.

Number of lots containing 1 defective tyres $= 10,000 \times 0.019604 = \mathbf{196}$ lots.

(iii) Probability of two defective tyres $P(2) = \dfrac{e^{-0.02}(0.02)^2}{2!} = 0.9802 \times 0.0002$

$= 0.00019604.$

Number of lots containing two defective tyres $= 10,000 \times 0.00019604 = 2$ **lots.**

23.14 Normal Distribution

Normal distribution is a continuous probability distribution. Any quantity whose variation depends on random causes will be distributed according to normal distribution.

Normal distribution can be derived from binomial distribution in the limiting case when the number of trial n is very large and the probability of success is close to $\frac{1}{2}$. The normal (or, Gaussian) distribution is defined as

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, \tag{23.40}$$

where the mean μ and the standard deviation $\sigma > 0$ are arbitrary constants. x is the normal variate and $f(x)$ is the probability density function of the normal distribution.

The graph of normal distribution is called the normal curve which is bell-shaped and symmetrical about the mean μ as shown in Fig. 23.5. The area under the normal curve between any two coordinates $x = x_1$ and $x = x_2$ is the probability of values falling into the given interval. The total area under the curve above the X-axis is

$$\int_{-\infty}^{\infty} f(x)dx = 1. \tag{23.41}$$

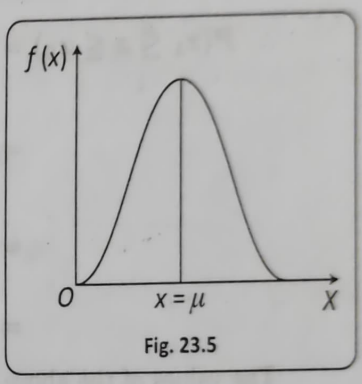

Fig. 23.5

N.B. $f(x)$ is called probability function, if

(i) $f(x) = 0$ **for every value of** x, (ii) $\displaystyle\int_{x_1}^{x_2} f(x)dx = P\ (x_1 < x < x_2)$

and (iii) $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1.$

Example 23.20. *A function* $f(x)$ *is defined as* $\begin{cases} 0, & x < 2 \\ \frac{1}{18}(2x+3), & 2 \le x \le 4 \\ 0, & x > 0. \end{cases}$

Show that it is a probability density function. **[AMIETE(Winter) 1994]**

Solution : Here $f(x) = 0, \quad x < 2$

$= \dfrac{1}{18}(2x+3), \quad 2 \le x \le 4$

$= 0, \quad x > 0.$

If $f(x)$ be a probability density function, then $\displaystyle\int_{-\infty}^{a} f(x)dx = 1.$

$\therefore \displaystyle\int_2^4 \frac{1}{18}(2x+3)dx = \frac{1}{18}[x^2+3x]_2^4 = \frac{1}{18}[16+12-4-6] = 1.$

$\therefore$ the given function is a probability density function.

23.14.1 Standard Form of Normal Distribution

The standard normal (or unit normal) distribution is characterised by a mean $\mu = 0$ and standard deviation $\sigma = 1$. It can be obtained by substituting the random variable $z = \frac{(x-\mu)}{\sigma}$ in equation

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

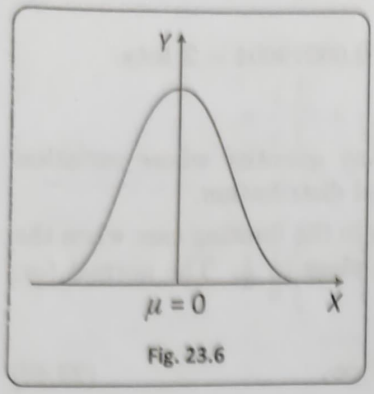

Fig. 23.6

Then the equation of normal distribution can be written as

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty. \tag{23.42}$$

The graph of the standard normal distribution is shown in Fig. 23.6.

(i) The probability of x lying between x_1 and x_2 is given by the area under the normal curve from x_1 to x_2.

$$\therefore \quad P(x_1 \le x \le x_2) = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz \quad \left[\because \ z = \frac{(x-\mu)}{\sigma} \ \text{ or, } \ dz = \frac{dx}{\sigma}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{z_2} e^{-\frac{z^2}{2}} dz - \int_0^{z_1} e^{-\frac{z^2}{2}} dz\right]$$

$$= P_2(z) - P_1(z). \tag{23.42}$$

The values of the above integrals are available in the table to the statistician.

(ii) The integral $\frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{z^2}{2}} dz$ is called the **probability integral** or **the error function represented by $f(z)$.**

23.14.2 Mean for Normal Distribution

$$\text{Mean} = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} x \, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} (t\sigma)(\sigma \, dt) \quad \left[\text{putting } t = \frac{x}{\sigma}\right]$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t e^{-\frac{t^2}{2}} dt = \frac{\sigma}{\sqrt{2\pi}} \left[e^{-\frac{t^2}{2}}\right]_{-\infty}^{\infty} = \frac{\sigma}{\sqrt{2\pi}}(0) = 0. \tag{23.43}$$

23.14.3 Standard Deviation for Normal Distribution

$$\mu_2' = \int x^2 f(x) dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx.$$

Put $\frac{x^2}{2a^2} = 1$ or, $x = \sqrt{2}\sigma t^{\frac{1}{2}}$, $dx = \frac{\sqrt{2}\sigma \, dx}{2t^{\frac{1}{2}}}$.

$$\therefore \mu_2' = \int_{-\infty}^{\infty} (2\sigma^2 t) \frac{1}{\sqrt{2\pi}\sigma} e^{-t\left(\frac{\sqrt{2}\sigma\, dt}{2t^{\frac{1}{2}}}\right)} = \frac{2\sigma^2}{\sigma\sqrt{2\pi}} \frac{\sqrt{2}\sigma}{2} \int_{-\infty}^{\infty} t^{\frac{3}{2}-1} e^{-t} dt$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \cdot 2 \cdot \Gamma\left(\frac{3}{2}\right), \quad \left[\because \int_0^{\infty} x^{n-1} e^{-x} dx = \Gamma(n)\right]$$

$$= 2\frac{\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi} = \sigma^2.$$

$$\therefore \mu_2 = \mu_2' - (\mu_1)^2 = \sigma^2 - (0)^2 = \sigma^2.$$

S.D. $= \sigma$. (23.44)

23.14.4 Variance and Standard Deviation for Normal Distribution

By definition of variance,

$$\text{Var}\,(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{z^2}{2}} \sigma\, dz, \quad \left[\text{putting } \frac{x-\mu}{\sigma} = z, \quad dx = \sigma\, dz\right]$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \cdot 2 \int_0^{\infty} z^2 e^{-\frac{z^2}{2}} dz = \frac{2\sigma^2}{\sqrt{2}} \frac{2(2)}{\sqrt{2}} \int_0^{\infty} te^{-t} t^{-\frac{1}{2}} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{\frac{3}{2}-1} dt, \quad \left[z = \sqrt{2}t^{\frac{1}{2}}, \ \therefore \ dz = \frac{1}{\sqrt{2}} t^{-\frac{1}{2}} dt\right]$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right), \quad \left[\because \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx\right]$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \sqrt{\pi} = \sigma^2. \tag{23.45}$$

$\therefore$ **Var $(x) = \sigma^2$ and σ is the standard deviation.**

Example 23.21. *In a normal distribution 31% of items are under 45 and 8% are over 64. Find the mean standard deviation of the distribution.*

[AMIE(Summer) 1991; WBUT 2003; JNTU 2003; PTU 2003; UPTU 2003]

Solution : Let the mean be m and σ, the standard deviation.

If 31% of the item are under 45, then the area to the left of the ordinate $x = 45$ is 0.31 (Fig. 23.7).

Assuming $z = z_1$ for $x = 45$, then

$$P(z_1 < z < 0) = 0.5 - 0.31 = 1.9.$$

From the tables, corresponding $P = 0.19$ is 0.5.

$\therefore \quad z = -0.5.$

When $x = 64$, $z = z_2$,

$$P(0 < z < z_2) = 0.5 - 0.08 = 0.42.$$

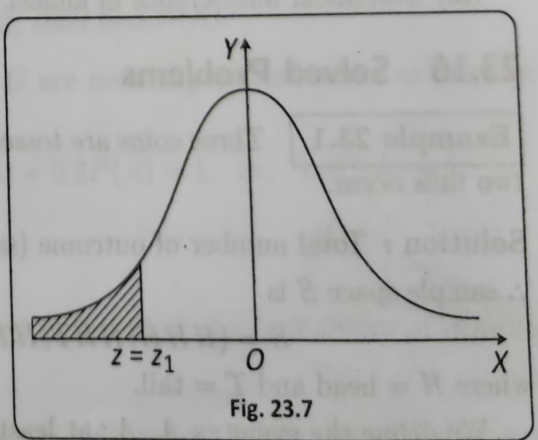

Fig. 23.7

Since 8% of the items are over 64, i.e., area to the right of the ordinate $x = 64$ is 0.08.
From the tables, the value of z corresponding to this area is 1.4.

$\therefore \quad z_2 = 1.4.$

Since, $z = \dfrac{x - m}{\sigma}$, $\therefore \quad -0.5 = \dfrac{45 - m}{\sigma}$ and $1.45 = \dfrac{64 - m}{\sigma}$.

Solving, we get $m = 50$ and $\sigma = 10$.

23.14.5 Table of Area $z = \frac{x}{\sigma}$

$z = \frac{x}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5570	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7697	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8907	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9102	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.2650	0.9279	0.9292	0.9316	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9409	0.9441
1.6	0.9452	0.9463	0.9474	0.9494	0.9495	0.9505	0.9515	0.9525	0.9505	0.9545

23.15 Applications of Normal Distributions

The normal distribution finds important applications in the following :

(i) Calculation of errors made by chance in experimental measurement

(ii) Computation of heat probability of a shot

(iii) Statistical interference in almost every branch of science.

23.16 Solved Problems

Example 23.1. *Three coins are tossed simultaneously. What is the probability that at least two tails occur.*

Solution : Total number of outcome (sample space) is elementary events.

$\therefore$ sample space S is

$$S = (HHH, HHT, HTH, THH, HTT, THT, TTH, TTT)$$

where H = head and T = tail.

We define the event as A. A : at least two tails occur.

Here, the favourable cases are HTT, THT, TTH and TTT.

$\therefore$ the probability that at least two tails occur, $P(A) = \frac{4}{8} = \frac{1}{2}$.

Example 23.2. *Calculate the probability of obtaining 4 heads in six tosses, using an unbiased coil.*

Solution : In this problem, $p = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 6$ and $r = 4$.

$$P(r) = {}^nC_r p^r q^{n-r} = \frac{6 \times 5}{1 \times 2}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2 = 15 \times \left(\frac{1}{2}\right)^6 = \frac{15}{64}.$$

Example 23.3. *The overall percentage of failures in an examination is 20%. If 6 candidates appear in the examination, what is the probability that at least 5 pass?*

Solution : By the problem, the probability of failures $q = \frac{20}{100} = \frac{1}{5}$.

$\therefore$ probability of pass $p = 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$.

$\therefore$ probability of at least 5 pass, i.e., 5 or 6 $= {}^6C_5 p^5 q + {}^6C_6 p^6 q^0 = 6\left(\frac{4}{5}\right)^5\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^6$

$$= \left(\frac{4}{5}\right)^6\left(\frac{6}{5} + \frac{4}{5}\right) = 2\left(\frac{4}{5}\right)^6 = \mathbf{0.6553}.$$

Example 23.4. *From a pack of well-shuffled cards, one card is drawn. Find the probability that this card is either a king or and ace.*

Solution : We define the events as A : card drawn is a king, B : card drawn is an ace.

The events A and B are mutually exclusive.

In a pack there are 52 cards, out of which 4 kings and 4 aces.

$\therefore$ $P(A) = \frac{4}{52} = \frac{1}{13}$, $\quad P(B) = \frac{4}{52} = \frac{1}{13}$.

As A and B are mutually exclusive, the probability,

P (either a king or an ace) $= P(A \cup B) = P(A) + P(B) = \frac{1}{13} + \frac{1}{13} = \mathbf{\frac{2}{13}}$.

Example 23.5. *If A, B, C are mutually exclusive and exhaustive events associated with a random experiment and $P(B) = 0.6$, $P(C) = 0.2P(A)$, then find $P(A)$.*

Solution : By the condition of the problem, A, B, C are mutually exclusive and exhaustive events. We have

$$P(A) + P(B) + P(C) = 1 \quad \text{or,} \quad P(A) + 0.6P(A) + 0.2P(A) = 1 \quad \text{or,} \quad 1.8P(A) = 1.$$

$\therefore$ $\mathbf{P(A) = \frac{1}{1.8} = \frac{5}{9}}$.

Example 23.6. *A bag contains 8 white and 6 red balls. Find the probability of drawing two balls of the same colour.*

Solution : 2 balls out of $8+6 = 14$ balls can be drawn in ${}^{14}C_2$ ways. This is the total outcome.

2 white balls out of 8 can be drawn in ${}^8C_2$ ways which is the total number of outcomes.

$\therefore$ probability of drawing 2 white balls out of $8 = \dfrac{{}^8C_2}{{}^{14}C_2} = \dfrac{28}{91}$.

Similarly, 2 red balls out of 6 can be drawn in ${}^6C_2$ ways.

$\therefore$ probability of drawing 2 red balls out of $6 = \dfrac{{}^6C_2}{{}^{14}C_2} = \dfrac{15}{91}$.

Hence, the probability of drawing 2 balls of the same colour (either both white or both red)

$$= \frac{28}{91} + \frac{15}{91} = \frac{43}{91}.$$

Example 23.7. *Two dice are tossed once. Find the probability of getting an even numbers on the first die or a total* 8.

Solution : Let us define the events as :

A : getting an even number on the first die, B : getting a total of 8

$\therefore$ the set of elementary events are as follows :

$A : [(2,1),\ldots,(2,6),(4,1)\cdots(4,6),(6.1),\ldots(6,6)]$, $\quad B : [(2,6),(3,5),(4,4),(5,3),(6,2)]$.

$A \cap B : [(2,6),(4,4),(6,2)]$

$\therefore$ $P(A) = \dfrac{10}{20} = \dfrac{1}{2}$, $\quad P(B) = \dfrac{5}{36}$ and $\quad P(A \cap B) = \dfrac{3}{36} = \dfrac{1}{12}$.

$\therefore$ P **(even number on the first die or a total of 8)** $= P \cup B = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{2} + \frac{5}{36} - \frac{1}{12} = \frac{5}{9}.$$

Example 23.8. *A box contains 2 white and 4 black balls. Another box B contains 5 white and 7 black balls. A ball is transferred from the box A to the box B. When a ball is drawn from the box B, find the probability that it is white.* **[VTU 2004; Cochin 1999]**

Solution : Here the required probability will depend on whether the transferred ball from box A to B is black or white.

If a black ball is transferred, then the probability $= \frac{4}{6}$.

After transference, box B will contain 5 white and 8 black balls. So, the probability of drawing a white ball from box B is $\frac{5}{13}$.

$\therefore$ the probability of drawing a white ball from B, if the transferred ball is black

$$= \frac{4}{6} \times \frac{5}{13} = \frac{10}{39}.$$

Similarly, the probability of drawing a white ball from B, if the transferred ball is white

$$= \frac{2}{6} \times \frac{6}{13} = \frac{2}{13}.$$

$\therefore$ **the required probability** $= \dfrac{10}{39} + \dfrac{2}{13} = \dfrac{16}{39}$.

Example 23.9. *In shuffling a pack of cards 4 are accidentally dropped. Find the chance that the missing cards should be one from each suit.*

Solution : Here the probability of 4 missing cards from different suits

$$= \frac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4} = \frac{13 \times 13 \times 13 \times 13 \times 4 \times 3 \times 2 \times 1}{52 \times 51 \times 50 \times 49}$$

$$= \frac{1}{4} \times \frac{13}{51} \times \frac{13}{50} \times \frac{13}{49} \times 24 = \frac{2197}{20825}.$$

Example 23.10. *The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that,*

(a) exactly two will be defective. (b) at least two will be defective. (c) none will be defective.

[BU(H) 2003; VTU 2004]

Solution : Given that the probability of defective pen $= \frac{1}{10} = 0.1$.

$\therefore$ the probability of non-defective pen $= 1 - 0.1 = \mathbf{0.9}$.

(a) The probability that exactly two pens will be defective $= {}^{12}C_2(0.1)^2(0.9)^{10} = \mathbf{0.2301}$.

(b) The probability that at least two pens will be defective

$= 1 - 1$(probability that none or one is non-defective)

$= 1 - [{}^{12}C_0(0.9)^{12} + {}^{12}C_1(0.1)(0.9)^{11}] = \mathbf{0.3412}$.

(c) The probability that none will be defective $= {}^{12}C_{12}(0.9)^{12} = \mathbf{0.2833}$.

Example 23.11. *Assume that on the average one telephone number out of fifteen called between 2 PM and 3 PM on weekdays is busy. What is the probability that if randomly selected telephone numbers are called (i) not more than three (ii) at least three of them will be busy.*

Solution : The probability of a telephone number being busy between 2 pm and 3 pm on weekdays is $p = \frac{1}{15}$.

$\therefore\ q = 1 - p = 1 - \frac{1}{15} = \frac{14}{15}$, $n = 6$.

$\therefore$ Binomial distribution $= \frac{14}{15} + \frac{1}{15}$.

(i) The probability that not more than three telephone numbers will be busy

$$= P(0) + P(1) + P(2) + P(3)$$

$$= {}^6C_0 \left(\frac{14}{15}\right)^6 \left(\frac{1}{15}\right)^0 + {}^6C_1 \left(\frac{14}{15}\right)^5 \left(\frac{1}{15}\right)^1 + {}^6C_2 \left(\frac{14}{15}\right)^4 \left(\frac{1}{15}\right)^2 + {}^6C_3 \left(\frac{14}{15}\right)^3 \left(\frac{1}{15}\right)^3$$

$$= \frac{14^3}{15^6}[2744 + 1176 + 210 + 20] = \frac{2744 \times 4150}{(15)^6} = \mathbf{0.9997}.$$

(ii) The probability that at least three of them will be busy

$$= P(3) + P(4) + P(5) + P(6)$$

$$= {}^6C_3 \left(\frac{14}{15}\right)^3 \left(\frac{1}{15}\right)^3 + {}^6C_4 \left(\frac{14}{15}\right)^2 \left(\frac{1}{15}\right)^4 + {}^6C_5 \left(\frac{14}{15}\right) \left(\frac{1}{15}\right)^5 + {}^6C_6 \left(\frac{1}{15}\right)^6$$

$$= \mathbf{0.005}.$$

Example 23.12. *A problem of physics is given to three students X, Y and Z having chances of solving it are $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$ respectively. What is the probability that the problem will be solved.*

Solution : The probability that X can solve the problem $= \frac{1}{3}$.

$\therefore$ the probability that X cannot solve the problem $= \left(1 - \frac{1}{3}\right)$.

Similarly, the probabilities that Y and Z cannot solve the problem are $\left(1 - \frac{1}{2}\right)$ and $\left(1 - \frac{2}{3}\right)$ respectively.

$\therefore$ the probability that X, Y, Z cannot solve the problem

$$= \left(1 - \frac{1}{3}\right) \times \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{2}{3}\right) = \frac{2}{3} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{9}.$$

Hence, the probability that the problem can be solved $= 1 - \frac{1}{9} = \frac{8}{9}$.

Example 23.13. *A box containing 10 articles is likely to have 2 defectives. In a consignment of 100 boxes, how many of them are expected to have three or less defectives?*

Solution : The probability that a box will contain defective articles is $p = \frac{2}{10} = \frac{1}{5}$.

The probability of non-defective articles is $q = (1 - p) = 1 - \frac{1}{5} = \frac{4}{5}$.

$\therefore$ the probability that box will contain three or less defective articles is

$$= P(0) + P(1) + P(2) + P(3)$$

$$= {}^{10}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10} + {}^{10}C_1 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^9 + {}^{10}C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8 + {}^{10}C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^7.$$

$$= 0.10738 + 0.2684 + 0.3020 + 0.2013 = 0.87908$$

$\therefore$ the number of boxes expected to contain three or less defective articles

$$= 0.87908 \times 100 \simeq \mathbf{88}.$$

Example 23.14. *In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails?* **[JNTU 2003]**

Solution : Here probability of head $P(H) = \frac{1}{2}$ and probability of tail $P(T) = \frac{1}{2}$.

According to binomial distribution, probability of 8 heads and 4 tails in 12 trials is

$$P(X = 8) = {}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4 = \frac{12!}{8!4!} \cdot \frac{1}{2^{12}} = \frac{495}{4096}.$$

Hence, expected number of such cases in 256 sets is

$$\boldsymbol{P(X = 8) \times 256 = \frac{495}{4096} \times 256 = 31.}$$

Example 23.15. *If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals (i) exactly 3 (ii) more than 2 individuals (iii) none (iv) more than one individual will suffer a bad reaction.*

[cf. VTU 2003]

Solution : Here the probability of occurrence is 0.001 which is very small. So Poisson's distribution is to be applied.

Here, $p = 0.001$, $n = 2000$.

$\therefore\ m = np = (2000) \times 0.001 = 2.$

$$P(r) = \frac{e^{-m} \cdot m_r}{r!} = \frac{e^{-2}(2)^r}{r!} = \frac{2^r}{r!e^2}.$$

(i) $P(3) = \frac{1}{3!}\frac{(2)^3}{e^2} = \frac{4}{3} \times (0.135) = \mathbf{0.18}.$

(ii) $P(>2) = P(3) + P(4) + \cdots + P(2000) = 1 - [P(0) + P(1) + P(2)]$

$$= 1 - \left[\frac{(2)^0}{0!e^2} + \frac{2^1 1!}{e^2} + \frac{2^2}{2!e^2}\right] = 1 - \frac{1}{e^2}[1 + 2 + 2] = 1 - \frac{5}{e^2}$$

$$= 1 - 5(0.135) = 1 - 0.675 = \mathbf{0.325}.$$

(iii) $P(\text{none}) = P(0) = \frac{e^{-2}2^0}{0!} = \mathbf{0.325}.$

(iv) $P(>1) = P(2) + P(3) + \cdots + P(2000) = 1 - [P(0) + P(1)]$

$$= 1 - \left[\frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!}\right] = 1 - \frac{3}{e^2} = 1 - 3(0.135)$$

$$= 1 - 0.405 = \mathbf{0.595}.$$

Example 23.16. *Using Poisson's distribution, find the probability that ace of spades will be drawn from a pack of well-shuffled cards at least once in 104 consecutive trials.*

Solution : For ace of spades, probability $P = \frac{1}{52}$, $n = 104$.

$$m = np = 104 \times \frac{1}{52} = 2$$

$$P(r) = \frac{e^{-m}m^r}{r!} = e^{-2} \cdot \frac{2^r}{r!} = \frac{1}{e^2} \cdot \frac{2^r}{r!}.$$

P (at least once) $= P(1) + P(2) + P(3) + \cdots + P(104) = 1 - P(0)$

$$= 1 - \frac{1}{e^2} \times \frac{2^0}{0!} = 1 - \frac{1}{e^2} = 1 - 0.135 = \mathbf{0.863}.$$

Example 23.17. *3% of bolts made by a machine are defective, the defect occurred at random during production. If bolts are packaged 50 per box, find (i) exact probability (ii) Poisson approximation to it, that a given box will contain 5 defectives.*

Solution : Here, $p = \frac{3}{100} = 0.03$.

(i) $q = 1 - p = 1 - 0.03 = 0.97.$

$\therefore$ probability of 5 defective balls in a lot of $50 = {}^{50}C_5(0.03)^5(0.97)^{45} = \mathbf{0.013074}.$

(ii) For Poisson distribution $m = np = 50 \times 0.03 = 1.5.$

$\therefore$ required Poisson distribution $P(r) = \frac{e^{-m}m^r}{r!} = \frac{e^{-1.5}(1.5)^5}{5!} = \mathbf{0.01412}.$

Example 23.18. *An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population what is the probability that not more than two of its clients one involved in such an accident next year? Given* $e^{-0.1} = 0.9048$ **[AMIETE(Summer) 1993]**

Solution : Here $p = 0.01\% = \frac{1}{100} \times \frac{1}{100} = \frac{1}{10000}$, $n = 1000$.

$\therefore \quad m = np = 1000 \times \frac{1}{10000} = \frac{1}{10} = 0.1.$

$P(r) = \frac{e^{-m}m^r}{r!}$.

P (not more than 2) $= P$ (0.1 and 2) $= P(0) + P(1) + P(2)$

$$= \frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^1}{1!} + \frac{e^{-0.1}(0.1)^2}{2!} = e^{-0.1}\left(1 + 0.1 + \frac{0.01}{2}\right)$$

$$= 0.9048 \times 1.05 = \mathbf{0.9998}.$$

Example 23.19. *A manufacturer knows that the razor blades he makes contain on the average 0.5% defectives. He packs them in packets of 5. What is the probability that a packet picked up at random will contain 3 or more faulty blades?*

Solution : Here $p = 0.5 = \frac{5}{100} = 0.005$, $n = 5$, $m = np = 5 \times 0.005 = 0.025$.

Then $P(r) = \frac{e^{-m}m^r}{r!}$.

$$\therefore \quad P(r \geq 3) = P(3) + P(4) + P(5) = \frac{e^{-0.025}(0.025)^3}{3!} + \frac{e^{-0.025}(0.025)^5}{4!} + \frac{e^{-0.025}(0.025)^5}{5!}$$

$$= \frac{e^{-0.025}}{5!}\frac{(0.025)^3}{5!}[20 + 5(0.025) + (0.025)^2]$$

$$= \frac{0.975 \times 0.000015625 \times 20.125625}{120} = \mathbf{0.000002555}.$$

Example 23.20. *A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days on which a car is not used and the proportion of days on which some demand is refused.* **[AMIE(Summer) 1995]**

Solution : Here $m = np = 1.5$.

(i) If the car is not used, then demand $(r) = 0$.

Then $P(r) = \frac{e^{-m}m^r}{r!}$.

$$\therefore \quad P(0) = \frac{e^{-m} \cdot m^0}{0!} = e^{-1.5} = \mathbf{0.223}.$$

(ii) Some demand is refused if the number of demands exceeds 2 (i.e., $r > 2$)

$$\therefore \quad P(r > 2) = P(3) + P(4) + \cdots = 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - \left[\frac{e^{-1.5}(1.5)^0}{0!} + \frac{e^{-1.5}(1.5)^1}{1!} + \frac{e^{-1.5}(1.5)^2}{2!}\right]$$

$$= 1 - e^{-1.5}(1 + 1.5 + 1.125) = 1 - 0.2231 \times 3.625$$

$$= 1 - 0.8087375 = \mathbf{0.1912625}.$$

Example 23.21. *A manufacturer produces airmail envelope whose weight is normally distributed with mean $\mu = 1.95$ g and standard deviation $\sigma = 0.05$. The envelopes are solid in lots of 1000. How many envelopes in a lot will be heavier than 2 g? Use the fact* $\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-\frac{z^2}{2}}dx = 0.3413$.

Solution : Given, $P(0 < z < 1) = \frac{1}{\sqrt{2\pi}}\int_0^1 e^{-\frac{z^2}{2}}dx = 0.3413$.

Also, $m = 1.95$, $\sigma = 0.05$ and $x = 2$.

$$\therefore\ z = \frac{x - m}{\sigma} = \frac{2 - 1.95}{0.05} = 1.$$

$\therefore\ P(z > 1)$ = area of the curve right to $z = 1 = 0.5 - 0.3413$.

Hence, number of envelopes heavier than 2 g in a lot of 1000 g is

$$N = 1000 \times 0.1587 = 158.7 \simeq \mathbf{159}.$$

Example 23.22. *In an examination the candidates are awarded the following grades depending on the marks scored by them :*

distinction :	$\geq 80\%$
second class :	$45\% \leq$ marks $\leq 80\%$
third class :	$30\% \leq$ marks $\leq 45\%$
fail :	$< 30\%$

It was found that 8% of the students failed and 8% have scored distinction. Find the average marks obtained by the candidates. Deduce the percentage of students placed in second class. Assume normal distribution of marks.

Solution : Suppose, the marks obtained by the candidates in the examination $= X$.

Let $X \sim N(m, \sigma^2)$.

We have, $P(X < 30) = 0.08$ and $P(X \geq 80) = 0.08$ as shown in Fig. 23.8.

The standard normal state is $Z = \frac{X - m}{\sigma}$.

Here, $X = 30$ is to the left of $X = m$ and $x = 80$ is to the right of $X = m$.

For $X = 30$, $z = \frac{30 - m}{\sigma} = -z_1$.

For $X = 80$, $z = \frac{80 - m}{\sigma} = z_2$.

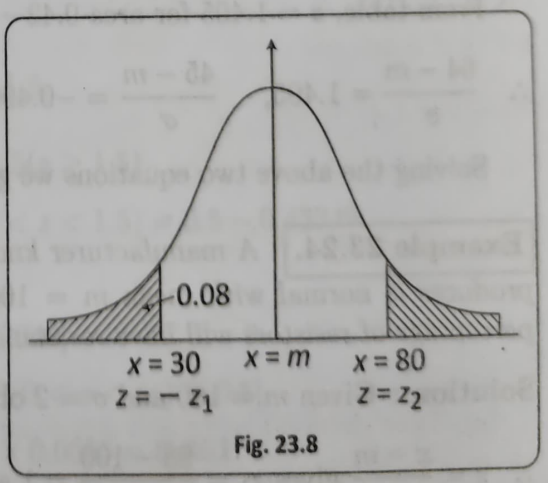

Fig. 23.8

$\because$ $P(X < 30) = 0.08$, we have

$$P(z < -z_1) \quad \text{or,} \quad P(0 < z < z_1) = 0.50 - 0.08 = 0.42.$$

From table we get $z_1 = 1.405 = z_2$.

$$\therefore \quad \frac{30 - m}{\sigma} = -1.405 \tag{i}$$

and $$\frac{80 - m}{\sigma} = 1.405. \tag{ii}$$

Adding Eqns. (i) and (ii), we get

$$110 - 2m = 0 \quad \text{or,} \quad m = 55$$

and $$\sigma = \frac{80 - m}{1.405} = \frac{25}{1.405} = \frac{25}{1.405} = \mathbf{17.79}.$$

If p be the probability that a candidate is placed in second class, then

$$P = P(45 < x < 60) = P(-0.56 \leq z < 0.28) = P(-0.56 \leq z < 0) + P(0 < z < 0.28)$$

$$= 0.2123 + 0.1103 = \mathbf{0.3226}.$$

Hence, approximately 32% candidates obtained second class.

Example 23.23. *In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and the standard deviation of the distribution.* [WBUT 2003]

Solution : If $x = 45$, then we have, $z = \dfrac{(x - m)}{\sigma} = \dfrac{45 - m}{\sigma}$,

where m = the mean and σ = the standard deviation of the distribution.

If $x = 46$, then $z = \dfrac{(64 - m)}{\sigma}$.

Now, the area between $z = 0$ and $z = \dfrac{(45 - m)}{\sigma} = (0.50 - 0.31) = 0.19$.

From table, $z = -0.496$ for area 0.19.

Area between $z = 0$ and $z = \dfrac{(64 - m)}{\sigma} = 0.50 - 0.08 = 0.42$.

From table, $z = 1.405$ for area 0.42

$$\therefore \quad \frac{64 - m}{\sigma} = 1.405, \qquad \frac{45 - m}{\sigma} = -0.496.$$

Solving the above two equations we get $\boldsymbol{m = 50}$ and $\boldsymbol{\sigma = 10}$.

Example 23.24. *A manufacturer knows from experience that the resistance of resistors he produces is normal with mean $m = 100$ ohms and standard deviation $\sigma = 2$ ohms. What percentage of resistors will have resistance between 98 ohms and 102 ohms?*

Solution : Given $m = 100$ and $\sigma = 2$ ohms; $x_1 = 98$ and $x_2 = 102$.

$$\therefore \quad z = \frac{x - m}{\sigma} \text{ gives } z_1 = \frac{98 - 100}{2} = 1 \text{ and } z_2 = \frac{102 - 100}{2} = +1.$$

Area between $z_1 = -1$ and $z_2 = +1$ [from (Fig. 23.9)]

= (area between $z = 0$ and $z = -1$)

+ (area between $z = 0$ and $z = +1$)

$= 2 \times$ (area between $z = 0$ and $z = +1$)

$= 2 \times 0.3413 = 0.6826.$

Hence, percentage of resistors having resistance between 98 ohms and 102 ohms = **68.26**.

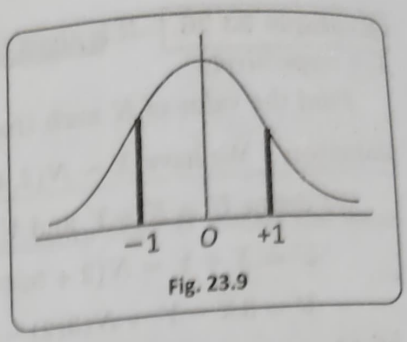

Fig. 23.9

Example 23.25. *A normal distribution has a mean 15.73 and a standard deviation of 2.08. Find the percentage of cases that fall between 17.81 and 13.6. Find also the percentage of cases lying above 18.85.*

Solution : In the problem, $m = 15.73$, $\sigma = 2.08$.

In the first case,

when $x_1 = 13.65$, $z_1 = \dfrac{x - m}{\sigma} = \dfrac{13.65 - 15.73}{2.08} = -\dfrac{2.08}{2.08} = 1$

and when $x_2 = 17.81$, $z_2 = \dfrac{17.81 - 15.73}{2.08} = -\dfrac{2.08}{2.08} = 1.$

$\therefore$ the required probability, $P(13.65 < x < 17.81) = P(0 < z_1 < -1) + P(0 < z_2 < 1)$

$= 0.34134 + 0.34134 = 0.68268.$

Hence, the required percentage $= 100 \times 0.68268 = \mathbf{68.268}$.

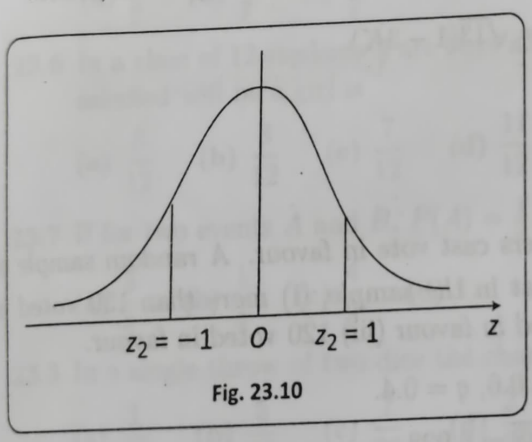

Fig. 23.10

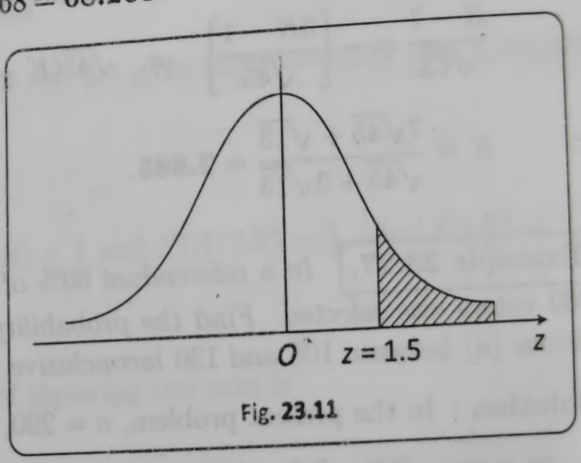

Fig. 23.11

Again, when $x = 18.85$, $z = \dfrac{18.85 - 15.73}{2.08} = \dfrac{3.12}{2.08} = 1.5.$

Probability $P(z > 15)$ = area of right to $z = 1.5 (z \geq 1.5)$

$= P(0 < z < \infty) - P(0 < z < 1.5) = 0.5 - 0.43349$

$= \mathbf{0.6651}.$

[$\because$ $z = 1.5$, the probability (from table) $= 0.43349$]

and [$\because$ the total area from the mean to the right $P(0 < z < \infty)$ is 0.5]

Hence, the percentage of cases above $18.85 = 100 \times 0.0665 = \mathbf{6.651}$.

Example 23.26. *It is given that X and Y are normal variates with means 2, 5 and variances 4, 9 respectively.*

Find the value of K such that, $P(X+Y \leq K) = P(3X - Y \leq 3K)$.

Solution : We have $X \sim N(2,4)$, $Y \sim N(5,9)$ and X, Y are independent.

We define $U = X + Y$ and $V = 3X - Y$. Then U and V are also normal variates.

$$U = X + Y \sim N(2+5, 4+9) = N(7,13)$$

$$V = 3X - Y \sim N(3(2) - 5, 9(4) + 9) = N(1,45).$$

Let us define $z = \dfrac{U-7}{\sqrt{13}}$.

Now, $z \sim N(0,1)$ and $U = K$,

$$z = \frac{V-1}{\sqrt{45}}.$$

Again, $z \sim N(0,1)$ and $V = 3K$.

$$z = \frac{3K-1}{\sqrt{45}} = z_2.$$

We require $P(X+Y \leq K) = P(3X - Y \geq 3K)$.

$$P(z \leq z_1) = P(z \geq z_2).$$

As the normal probability curve is symmetric about $z = 0$, this implies that $z_1 = -z_2$.

$$\therefore \quad \frac{K-7}{\sqrt{13}} = -\left[\frac{3K-1}{\sqrt{45}}\right] \quad \text{or,} \quad \sqrt{45}(K-7) = \sqrt{13}(1-3K).$$

$$\therefore \quad K = \frac{7\sqrt{45} + \sqrt{13}}{\sqrt{45} + 3\sqrt{13}} = \mathbf{2.885}.$$

Example 23.27. *In a referendum 60% of voters cast vote in favour. A random sample of 200 voters was selected. Find the probability that in the sample,* (i) *more than 130 voted in favour* (ii) *between 105 and 130 inconclusive voted in favour* (iii) *120 voted in favour.*

Solution : In the present problem, $n = 200$, $p = 0.6$, $q = 0.4$.

$$m = np = 200 \times 0.6 = 120, \qquad \sigma = \sqrt{npq} = \sqrt{48} = 6.928.$$

(i) $$P(X > 130) = P(x > 130.5) = P\left(X > \frac{130.5 - 120}{\sqrt{48}}\right) = P(z > 1.516) = \mathbf{0.0648}.$$

(ii) $$P(105 < x < 130) = P(105.5 < x < 129.5) = P\left(\frac{105.5 - 120}{\sqrt{48}} < z < \frac{129.5 - 120}{\sqrt{48}}\right)$$

$$= P(-2.09 < z < 1.37) = \mathbf{0.8964}.$$

(iii) $$P(x = 120) = P(119.5 < x < 120.5) = P(-0.072 < z < 0.072) = \mathbf{0.0575}.$$